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AN ASYMPTOTIC ANALYSIS OF THE GENTLE DYNAMIC LOADING OF AN ELASTIC PLATE†

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Abstract—The equations of the approximate theories for the dynamics of thin elastic plates are obtained by applying formal asymptotic expansion techniques to the exact three dimensional theory of dynamic elasticity. The usual boundary conditions of the approximate theories are derived from a boundary layer and diffusion layer analysis.

1. INTRODUCTION

ONE of the most satisfactory derivations of the classical equations governing the static deflection of a thin plate is that due to Friedrichs and Dressler [2]. Starting with the three-dimensional linear theory of elasticity for an isotropic homogeneous medium these authors use formal perturbation expansions in powers of the plate thickness to solve a boundary value problem for a plate. They obtain an "interior" expansion whose leading term satisfies the classical plate equations and with later terms giving "thick" plate corrections. In addition a "boundary layer" correction important near the edge of the plate is added to provide a uniformly valid expansion. The investigation of this boundary layer correction also provides a systematic derivation of suitable boundary conditions to use with the differential equations governing each term of the interior expansion.

The present work uses the technique described above to investigate a dynamic problem for a thin elastic plate. The word "gentle" in the title refers to the fact that only moderate rates of loading (defined explicitly in Section 2) are considered. In the dynamic case expansions analogous to the interior and boundary layer expansions of the static problem again occur. However additional "diffusion" and "long-time" interior expansions must also be added to construct a composite expansion which is uniformly valid in both space and time. These three additional expansions are required to describe the diffusion of bending effects away from the edge of the plate and the relatively slow build up of bending vibrations. The various classical equations governing the transverse and extensional vibration of a thin plate (see Love [3, pp. 496–497]) and the associated boundary conditions all arise naturally in the systematic development of the expansions mentioned above.

In the development of this work the four different expansions were investigated concurrently and some procedures followed depend on the observation of the interaction of the various expansions. For this exposition the expansions are considered consecutively and therefore certain procedures are temporarily justified by reference to a "preliminary investigation" and then fully justified by the results of a subsequent section.

† Part of this work is based on the author's doctoral dissertation [1] presented in 1968 to the Division of Engineering and Applied Science, California Institute of Technology.

The methods used here are formal in nature; it remains to be proved that the composite expansion obtained is a valid approximation for the exact solution of the problem considered.

The recent derivation of various asymptotic dynamic theories for cylindrical shells by Johnson and Widera [4] uses methods similar to those applied here.

2. FORMULATION OF THE PROBLEM

An initial-boundary value problem for an elastic plate with a minimum diameter $2R$ and thickness $2h$ is considered. The fact that the plate is thin means that the parameter $\varepsilon = h/R$ is small. In order to exhibit the dependence of the problem on ε explicitly we introduce the nondimensional independent variables x, y, z, t defined by

$$x = \frac{X}{R}, \qquad y = \frac{Y}{R}, \qquad z = \frac{Z}{h}, \qquad t = \frac{c_2 T}{R}, \tag{2.1}$$

where X, Y, Z, T represent the physical coordinate system (with the midplane of the plate in the XY-plane), and c_2 is the shear wave velocity for the material of the plate. Nondimensional dependent variables are also introduced: all stresses are divided by the Lamé shear modulus μ for the plate material and all displacements are divided by the half thickness h. Now our problem can be stated in the following dimensionless form. We wish to find the displacement vector $\mathbf{u}(x, y, z, t; \varepsilon)$, with components u, v, w, satisfying the equations of motion

$$\frac{\partial \tau_{xz}}{\partial z}+\varepsilon\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}\right)-\varepsilon^2\frac{\partial^2 u}{\partial t^2} = 0, \tag{2.2a}$$

$$\frac{\partial \tau_{yz}}{\partial z}+\varepsilon\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}\right)-\varepsilon^2\frac{\partial^2 v}{\partial t^2} = 0, \tag{2.2b}$$

$$\frac{\partial \sigma_z}{\partial z}+\varepsilon\left(\frac{\partial \tau_{xz}}{\partial x}\quad\frac{\partial \tau_{yz}}{\partial y}\right)-\varepsilon^2\frac{\partial^2 w}{\partial t^2} = 0, \tag{2.2c}$$

for $x, y \varepsilon C$, $-1 < z < 1$ and $0 < t < \infty$. Here C is an open, simply connected region in the xy-plane bounded by a smooth curve Γ. The stress components $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}$ appearing in equations (2.2a, b, c) are related to the displacement components as follows:

$$\sigma_x = (\alpha-2)\frac{\partial w}{\partial z}+\varepsilon\left(\alpha\frac{\partial u}{\partial x}+(\alpha-2)\frac{\partial v}{\partial y}\right), \tag{2.3a}$$

$$\sigma_y = (\alpha-2)\frac{\partial w}{\partial z}+\varepsilon\left[(\alpha-2)\frac{\partial u}{\partial x}+\alpha\frac{\partial v}{\partial y}\right], \tag{2.3b}$$

$$\sigma_z = \alpha\frac{\partial w}{\partial z}+\varepsilon(\alpha-2)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \tag{2.3c}$$

$$\tau_{xy} = \varepsilon\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \tag{2.3d}$$

$$\tau_{yz} = \frac{\partial v}{\partial z} + \varepsilon \frac{\partial w}{\partial y}, \tag{2.3e}$$

$$\tau_{xz} = \frac{\partial u}{\partial z} + \varepsilon \frac{\partial w}{\partial x}, \tag{2.3f}$$

where $\alpha = c_1^2/c_2^2$ and c_1 is the dilation wave velocity. Before prescribing the boundary conditions we must explicitly define what is meant by the word "gentle" in the title: if the time scale associated with the prescribed loading is comparable to the time required for a shear wave to travel across the minimum diameter of the plate then we say that the loading is gentle. In this work we assume that all prescribed stresses are gentle and thus the use of the time variable t as defined in equation (2.1) follows naturally.

The prescribed boundary conditions on the faces of the plate are

$$\begin{aligned} \sigma_z(x, y, 1, t; \varepsilon) = \varepsilon^4 p_1^{(4)}(x, y, t), \qquad \sigma_z(x, y, -1, t; \varepsilon) = \varepsilon^4 p_2^{(4)}(x, y, t), \\ \tau_{yz}(x, y, \pm 1, t; \varepsilon) = 0, \qquad \tau_{xz}(x, y, \pm 1, t; \varepsilon) = 0, \end{aligned} \tag{2.4}$$

for $x, y \varepsilon \bar{C}$, $0 \leq t < \infty$, where $\bar{C}$ is the closure of C. Here and in what follows a superscript always matches an associated power of ε. The selection of the order of the boundary condition above as $O(\varepsilon^4)$ is not expected to be obvious at this stage. However, to be consistent with the small displacement assumption of linear elasticity we only wish to consider displacements uniformly of $O(1)$ or smaller. It turns out that direct stresses of $O(\varepsilon^4)$ on the faces of the plate are the largest possible stresses producing the required $O(1)$ displacements. These remarks also hold for the other boundary conditions given below. It should be noted that since the problem is linear a change in the order of the prescribed boundary conditions will be accompanied by the same change in the order of the solution for the displacements. Therefore a problem with boundary conditions of some other order in ε can be solved simply by multiplying the solutions for the displacements obtained below by the appropriate power of ε followed by the corresponding change in superscripts. Still more general boundary conditions in the form of a polynomial in ε or a power series in ε can be solved by superposition.

The boundary conditions on the edge of the plate are:

$$\sigma_n = \varepsilon^2 f^{(2)}(x, y, z, t), \qquad \tau_{ns} = \varepsilon^2 g^{(2)}(x, y, z, t), \qquad \tau_{nz} = \varepsilon^3 h^{(3)}(x, y, z, t) \tag{2.5}$$

for $x, y \varepsilon \Gamma$, $-1 \leq z \leq 1$, $0 \leq t < \infty$. Here n refers to the direction normal to Γ and s refers to the direction tangential to Γ.

General initial displacements lead to the propagation of waves which multiply reflect at the faces of the plate. Since we wish to avoid such small scale propagation effects in this work we assume initial quiescence. That is:

$$\mathbf{u}(x, y, z, 0; \varepsilon) = 0, \qquad \frac{\partial \mathbf{u}}{\partial t}(x, y, z, 0; \varepsilon) = 0, \tag{2.6}$$

for $x, y \varepsilon \bar{C}$, $-1 \leq z \leq 1$.

In later sections the stretching and bending effects require different treatment. In fact we find that the bending effects occur with two time scales. To take account of this fact we must separate the long-time behaviour of the prescribed boundary stresses from their short-time behaviour. We assume the alternative representations for the data in the form

$$p_1^{(4)}(x, y, t) = \bar{p}_1^{(4)}(x, y, t) + \hat{p}_1^{(4)}(x, y, \varepsilon t) \tag{2.7}$$

with similar results for $p_2^{(4)}$, $f^{(2)}$, $g^{(2)}$ and $h^{(3)}$. Here the term $\bar{p}_1^{(4)}$ represents the short-time effects and $\hat{p}_1^{(4)}$ represents the long-time effects. The properties required of the data, such as smoothness and asymptotic behaviour, are discussed when they become important in the later sections. At present we just prescribe that the data are sufficiently smooth to allow solutions for u, v, w of (2.2)–(2.6) which are at least twice continuously differentiable.

3. INTERIOR EXPANSION

Following the procedure of Friedrichs and Dressler we seek a solution $\mathbf{u} = \mathbf{u}_i$ where $\mathbf{u}_i$, the interior expansion, has the form

$$\mathbf{u}_i(x, y, z, t; \varepsilon) = \varepsilon \mathbf{u}_i^{(1)}(x, y, z, t) + \varepsilon^2 \mathbf{u}_i^{(2)}(x, y, z, t) + \cdots. \tag{3.1}$$

The choice of the order of the leading term in the interior expansion as $O(\varepsilon)$, based on a preliminary investigation, avoids some unnecessary algebra, but is not fundamental to the method. We could equally well assume a leading term of $O(1)$ and deduce the result $\mathbf{u}_i^{(0)} = 0$. We now substitute the expansion (3.1), and corresponding expansions for the stress components, into the equations (2.2)–(2.6) and solve the problem associated with each power of ε sequentially. From the leading terms in the equations of motion (2.2a, b, c) we obtain

$$\frac{\partial \tau_{xzi}^{(1)}}{\partial z} = 0, \qquad \frac{\partial \tau_{yzi}^{(1)}}{\partial z} = 0, \qquad \frac{\partial \sigma_{zi}^{(1)}}{\partial z} = 0, \tag{3.2}$$

where as above the subscript i refers to quantities associated with the interior expansion. In general it is not possible to satisfy all the prescribed boundary conditions with solutions of (3.2) however by ignoring the edge conditions temporarily (as in the static case, see [2]) we can proceed. The general solutions of (3.2) are independent of z and in order to satisfy the prescribed boundary conditions

$$\tau_{xzi}^{(1)}(x, y, \pm 1, t) = \tau_{yzi}^{(1)}(x, y, \pm 1, t) = \sigma_{zi}^{(1)}(x, y, \pm 1, t) = 0,$$

we must take $\tau_{xzi}^{(1)} = \tau_{yzi}^{(1)} = \sigma_{zi}^{(1)} \equiv 0$. Then from the stress displacement equations (2.3c, e, f) we obtain $u_i^{(1)} = U_i^{(1)}$, $v_i^{(1)} = V_i^{(1)}$ and $w_i^{(1)} = W_i^{(1)}$ where $U_i^{(1)}$, $V_i^{(1)}$ and $W_i^{(1)}$ are arbitrary functions of x, y and t only.

Applying the same procedure to the $O(\varepsilon^2)$ equations of motion and boundary conditions we obtain the following formulae for the $O(\varepsilon^2)$ displacements

$$u_i^{(2)} = -z\frac{\partial W_i^{(1)}}{\partial x} + U_i^{(2)},$$

$$v_i^{(2)} = -z\frac{\partial W^{(2)}}{\partial y} + V_i^{(2)},$$

$$w_i^{(2)} = -z\frac{\alpha-2}{\alpha}\Delta_i^{(1)} + W_i^{(2)}.$$

Here $U_i^{(2)}$, $V_i^{(2)}$ and $W_i^{(2)}$ are arbitrary functions of x, y and t, and $\Delta_i^{(1)}$, the average dilation, is defined by the equation

$$\Delta_i^{(1)} = \frac{\partial U_i^{(1)}}{\partial x}+\frac{\partial V_i^{(1)}}{\partial y}.$$

Using the results from the $O(\varepsilon)$ and $O(\varepsilon^2)$ problems the $O(\varepsilon^3)$ equations of motion become:

$$\frac{\partial \tau_{xzi}^{(3)}}{\partial z}+F_x = 0, \qquad \frac{\partial \tau_{yzi}^{(3)}}{\partial z}+F_y = 0, \qquad \frac{\partial \sigma_{zi}^{(3)}}{\partial z}+F_z = 0,$$

where F_x, F_y and F_z are functions of x, y and t only defined below. The $O(\varepsilon^3)$ term of the boundary conditions (2.4) can only be satisfied, giving solutions $\tau_{xzi}^{(3)} = \tau_{yzi}^{(3)} = \sigma_{zi}^{(3)} \equiv 0$, provided that the following restrictions on F_x, F_y and F_z are imposed:

$$F_x = \frac{4(\alpha-1)}{\alpha}\frac{\partial^2 U_i^{(1)}}{\partial x^2}+\frac{\partial^2 U_i^{(1)}}{\partial y^2}-\frac{\partial^2 U_i^{(1)}}{\partial t^2}+\frac{3\alpha-4}{\alpha}\frac{\partial^2 V_i^{(1)}}{\partial x\partial y} = 0, \tag{3.3a}$$

$$F_y = \frac{3\alpha-4}{\alpha}\frac{\partial^2 U_i^{(1)}}{\partial x\partial y}+\frac{\partial^2 V_i^{(1)}}{\partial x^2}+\frac{4(\alpha-1)}{\alpha}\frac{\partial^2 V_i^{(1)}}{\partial y^2}-\frac{\partial^2 V_i^{(1)}}{\partial t^2} = 0, \tag{3.3b}$$

and

$$F_z = \frac{\partial^2 W_i^{(1)}}{\partial t^2} = 0. \tag{3.3c}$$

The initial quiescence condition (2.6) is satisfied at this order by taking

$$U_i^{(1)} = V_i^{(1)} = W_i^{(1)} = \frac{\partial U_i^{(1)}}{\partial t} = \frac{\partial V_i^{(1)}}{\partial t} = \frac{\partial W_i^{(1)}}{\partial t} = 0, \tag{3.4}$$

at $t = 0$. These conditions are sufficient to determine the solution of (3.3c) as $W_i^{(1)} \equiv 0$. However, boundary conditions as well as initial conditions are required for the solution of (3.3a, b). In general the prescribed boundary data (2.5) depend on the thickness coordinate z and therefore cannot be used to determine $U_i^{(1)}$ and $V_i^{(1)}$. The boundary layer analysis of Section 5 leads to the conclusion that the interior expansion need only take account of the thickness average of the prescribed stresses. Anticipating this result we take

$$\begin{aligned}\sigma_{ni}^{(2)} &= \frac{1}{2}\int_{-1}^{1} f^{(2)}(x, y, z, t)\,\mathrm{d}z,\\ \tau_{nsi}^{(2)} &= \frac{1}{2}\int_{-1}^{1} g^{(2)}(x, y, z, t)\,\mathrm{d}z,\end{aligned} \tag{3.5}$$

for $x, y \varepsilon \Gamma$ and $0 \le t < \infty$. These conditions, together with the initial conditions (3.4), fully determine $U_i^{(1)}$ and $V_i^{(1)}$.

By differentiating (3.3a) with respect to x, (3.3b) with respect to y and adding we obtain the equation

$$\frac{4(\alpha-1)}{\alpha}\nabla^2\Delta_i^{(1)}-\frac{\partial^2\Delta_i^{(1)}}{\partial t^2} = 0, \tag{3.6}$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. Equation (3.6) can readily be identified as the plate wave equation (see Love [3, p. 497]) when rewritten in terms of physical variables. Similarly by differentiating (3.3a) with respect to y, (3.3b) with respect to x and subtracting we obtain

$$\nabla^2 \Omega_i^{(1)} - \frac{\partial^2 \Omega_i^{(1)}}{\partial t^2} = 0, \tag{3.7}$$

where $\Omega_i^{(1)} = \partial U_i^{(1)}/\partial y - \partial V_i^{(1)}/\partial x$ is the average rotation. Equation (3.7) is the classical equation governing the torsional stretching of a plate (see Love [3, p. 497]). Thus the leading terms in the interior expansion describe the propagation of the average stretching effects (dilatational and torsional) into the interior of the plate.

Proceeding to the higher orders in equations (2.2)–(2.4) we can obtain successive terms in the expansion for $\mathbf{u}_i$. All terms have the form of a polynomial in the thickness coordinate z with coefficients depending on x, y and t (details of the expansion and the corresponding expansions for the stresses are contained in [1]). At each order the terms independent of z, written as $U_i^{(k)}$, $V_i^{(k)}$ and $W_i^{(k)}$, $k = 1, 2, \ldots$, are required to satisfy equations similar to (3.3a, b, c). The important fact about the higher order equations corresponding to (3.3a, b) is that their right hand sides contain successively higher derivatives of the data $p_1^{(4)}$ and $p_2^{(4)}$ and also derivatives of the lower order terms $U_i^{(k-2)}$, $V_i^{(k-2)}$, $U_i^{(k-4)}$, $V_i^{(k-4)}$, ... Therefore we can only calculate as many terms in the inner expansion as the smoothness of $p_1^{(4)}$, $p_2^{(4)}$, $f^{(2)}$ and $g^{(2)}$ allow. In fact it can be shown that the equations for $U_i^{(k)}$ and $V_i^{(k)}$, $k = 1, 2, \ldots, N$ all have twice continuously differentiable solutions provided that $p_1^{(4)}$ and $p_2^{(4)}$ have $N-2$ continuous derivatives uniformly of $0(1)$ in comparison with ε, and provided that $f^{(2)}$ and $g^{(2)}$ have N continuous derivatives uniformly of $0(1)$ in comparison with ε.

It is also easily shown that the solutions for $W_i^{(2k)}$, $k = 1, 2, \ldots$, involve $2k$ repeated integrals whose integrand consists of various $(2k-2)$th order derivatives of $p_1^{(4)} - p_2^{(4)}$. Therefore, for this part of the interior expansion we consider the short time effects only and use $\bar{p}_1^{(4)} - \bar{p}_2^{(4)}$ instead in the solutions for $W_i^{(2k)}$. Here $\bar{p}_1^{(4)}$ and $\bar{p}_2^{(4)}$ have the same smoothness properties as those prescribed above for $p_1^{(4)}$ and $p_2^{(4)}$ and in addition $\bar{p}_1^{(4)}, \bar{p}_2^{(4)} \sim 0(1/t^{2N+1})$ as $t \to \infty$. This decay requirement avoids the non-uniformly valid long-time behaviour of w_i as a $2N+1$th order polynomial in εt otherwise predicted by the higher order equations corresponding to (3.3c).

4. LONG-TIME INTERIOR EXPANSION

With the modification of the interior expansion to describe short-time bending effects only it now satisfies the boundary condition

$$\sigma_{zi}(x, y, \pm 1, t; \varepsilon) = \varepsilon^4[\tfrac{1}{2}(p_1^{(4)} + p_2^{(4)}) \pm \tfrac{1}{2}(\bar{p}_1^{(4)} - \bar{p}_2^{(4)})].$$

To investigate the remaining long-time bending effects we introduce a new time variable τ defined by the scaling $\tau = \varepsilon t$: a choice suggested by the discussion at the end of the previous section. We now seek a solution of the problem (2.2)–(2.6) in the form $\mathbf{u} = \mathbf{u}_i(x, y, z, t; \varepsilon) + \mathbf{u}_l(x, y, z, \tau; \varepsilon)$, where $\mathbf{u}_l$ is called the long-time interior approximation (a subscript l always indicates an association with this approximation). Now substituting in (2.2)–(2.6) and using the properties of the interior expansion we obtain the governing equations

$$\frac{\partial \tau_{xzl}}{\partial z}+\varepsilon\left(\frac{\partial \sigma_{xl}}{\partial x}+\frac{\partial \tau_{xyl}}{\partial y}\right)-\varepsilon^4\frac{\partial^2 u_l}{\partial \tau^2}=0, \tag{4.1a}$$

$$\frac{\partial \tau_{yzl}}{\partial z}+\varepsilon\left(\frac{\partial \tau_{xyl}}{\partial x}+\frac{\partial \sigma_{yl}}{\partial y}\right)-\varepsilon^4\frac{\partial^2 v_l}{\partial \tau^2}=0, \tag{4.1b}$$

$$\frac{\partial \sigma_{zl}}{\partial z}+\varepsilon\left(\frac{\partial \tau_{xzl}}{\partial x}+\frac{\partial \tau_{yzl}}{\partial y}\right)-\varepsilon^4\frac{\partial^2 w_l}{\partial \tau^2}=0. \tag{4.1c}$$

Note that equations (4.1a, b, c) are only valid up to the order at which the interior expansion is truncated. The relationship between stress components and displacement components is unchanged by a scaling in the time variable and therefore equations (2.3a–f) also apply here.

On the faces of the plate the boundary conditions are

$$\begin{aligned}\sigma_{zl}(x,y,\pm 1,\tau;\varepsilon) &= \pm\frac{\varepsilon^4}{2}(\hat{p}_1^{(4)}-\hat{p}_2^{(4)}),\\ \tau_{xzl}(x,y,\pm 1,\tau;\varepsilon) &= \tau_{yzl}(x,y,\pm 1,\tau;\varepsilon)=0,\end{aligned} \tag{4.2}$$

for $x, y\varepsilon\bar{C}$ and $0 \le \tau < \infty$. On the edges of the plate

$$\begin{aligned}\sigma_{nl} &= \varepsilon^2 f^{(2)}-\sigma_{ni}, \qquad \tau_{ns} = \varepsilon^2 g^{(2)}-\tau_{nsi},\\ \tau_{nzl} &= \varepsilon^3 h^{(3)}-\tau_{nzi},\end{aligned} \tag{4.3}$$

for $x, y\varepsilon\Gamma$, $-1 \le z \le 1$ and $0 \le \tau < \infty$. Initial conditions are also required to complete the problem. These are derived from the analysis of the diffusion expansion of the next section and will be discussed later.

We now assume a solution for $\mathbf{u}_l$ in the form of a power series in ε; $\mathbf{u}_l = \mathbf{u}_l^{(0)}+\varepsilon\mathbf{u}_l^{(1)}+\dots$, and following procedure of Section 3 solve for successive terms. The leading terms are (higher order terms are given in [1]):

$$\begin{aligned}u_l &= -\varepsilon z\frac{\partial W_l^{(0)}}{\partial x}+\cdots, \qquad v_l = -\varepsilon z\frac{\partial W_l^{(0)}}{\partial y}+\cdots,\\ w_l &= W_l^{(0)}+\cdots.\end{aligned} \tag{4.4}$$

The function $W_l^{(0)}$ depends on x, y and τ only and is required to satisfy the equation

$$\frac{4(\alpha-1)}{3\alpha}\nabla^4 W_l^{(0)}+\frac{\partial^2 W_l^{(0)}}{\partial \tau^2} = \tfrac{1}{2}(\hat{p}_1^{(4)}-\hat{p}_2^{(4)}). \tag{4.5}$$

When rewritten in terms of physical variables equation (4.5) can be recognized as the classical equation of the approximate bending vibration of a thin elastic plate (see Love [3, p. 497]).

Suitable initial conditions and boundary conditions to be used with equation (4.5) are derived in Sections 5 and 6, respectively.

When higher order terms are obtained in the expansions (4.4) it is found that w_l is even in z corresponding to a purely bending type of deflection. This is to be expected since the interior expansion developed in Section 3 describes the average stretching effects with uniform validity in time. Higher order terms in the expansion for w_l involve functions of x, y and τ, $W_l^{(k)}$ say, satisfying equations similar to (4.5) with right hand sides depending on derivatives of $\hat{p}_1^{(4)}$ and $\hat{p}_2^{(4)}$. It is found that with $\hat{p}_1^{(4)}$ and $\hat{p}_2^{(4)}$ satisfying the same smoothness conditions as $p_1^{(4)}$ and $p_2^{(4)}$ the long-time interior expansion and the interior expansion truncate at the same order.

5. DIFFUSION EXPANSION

The combined expansion $\mathbf{u}_i+\mathbf{u}_l$ obtained in the previous two sections satisfies the equations of motion (2.2), the initial conditions (2.6) and the boundary conditions on the faces of the plate (2.4) [all up to $O(\varepsilon^N)$ with N determined by the smoothness of the prescribed data]. However, in general the boundary conditions on the edge of the plate are not satisfied. In the static case Friedrichs and Dressler [2] described the average stretching and bending effects with an interior expansion and found that the nature of the distribution of the prescribed edge stresses through the thickness of the plate only influences a region near the edge called the boundary layer. Therefore, for a static problem an interior expansion and a boundary layer expansion are together sufficient to give a uniformly valid approximation to the solution. In the dynamic case the bending moment diffuses away from the edge quite slowly. The long-time interior expansion of Section 4 can be used to describe the latter stage of this diffusion process, however some other expansion is required to describe the initial stage. A preliminary analysis showed that this initial diffusion of bending moment could not be contained in a region as small as the static boundary layer which is $0(\varepsilon)$ in thickness, and suggested a scale of $0(\varepsilon^{\frac{1}{2}})$ instead. Therefore we introduce the variables s and $\rho = n/\varepsilon^{\frac{1}{2}}$ where s measures arc length around Γ and n measures distance along a normal to Γ.

We now try a solution u in the form of a composite expansion

$$u = u_i(x, y, z, t; \varepsilon)+u_l(x, y, z, \tau; \varepsilon)+u_d(\rho, s, z, t; \varepsilon), \tag{5.1}$$

where u_d is called the diffusion approximation (a subscript d always refers to this approximation). The equations of motion, in terms of the displacement components u_{nd}, u_{sd}, u_{zd} and the stress components σ_{nd}, σ_{sd}, σ_{zd}, τ_{nsd}, τ_{nzd}, τ_{szd} are:

$$\frac{\partial \tau_{nzd}}{\partial z}+\varepsilon^{\frac{1}{2}}\frac{\partial \sigma_{nd}}{\partial \rho}+\frac{\varepsilon}{r}\left[\frac{\partial \tau_{nsd}}{\partial s}+\frac{1}{a}(\sigma_{sd}-\sigma_{nd})\right]-\varepsilon^2\frac{\partial^2 u_{nd}}{\partial t^2} = 0, \tag{5.2a}$$

$$\frac{\partial \tau_{szd}}{\partial z}+\varepsilon^{\frac{1}{2}}\frac{\partial \tau_{nsd}}{\partial \rho}+\frac{\varepsilon}{r}\left(\frac{\partial \sigma_{sd}}{\partial s}-\frac{2}{a}\tau_{nsd}\right)-\varepsilon^2\frac{\partial^2 u_{sd}}{\partial t^2} = 0, \tag{5.2b}$$

$$\frac{\partial \sigma_{zd}}{\partial z}+\varepsilon^{\frac{1}{2}}\frac{\partial \tau_{nzd}}{\partial \rho}+\frac{\varepsilon}{r}\left(\frac{\partial \tau_{szd}}{\partial s}-\frac{1}{a}\tau_{nzd}\right)-\varepsilon^2\frac{\partial^2 u_{zd}}{\partial t^2} = 0, \tag{5.2c}$$

for $0 < \rho < \infty$, $0 \le s \le L$, $-1 < z < 1$ and $0 < t < \infty$, where L is the length of Γ, $a(s)$ is the radius of curvature of Γ and $r = (1-\varepsilon^{\frac{1}{2}}\rho/a)$. Note that the actual finite but large domain for ρ has been replaced by the approximate semi-infinite domain. Since we expect the bending effects considered here to be unimportant away from the edge this appears to be a reasonable approximation (a similar approximation is made in the static boundary layer analysis of [2]). In fact, as is discussed later, this approximation is only valid for a limited time.

The equations relating stresses and displacements are:

$$\sigma_{nd} = (\alpha-2)\frac{\partial u_{zd}}{\partial z}+\varepsilon^{\frac{1}{2}}\frac{\partial u_{nd}}{\partial \rho}+\varepsilon\frac{\alpha-2}{r}\left(\frac{\partial u_{sd}}{\partial s}-\frac{1}{a}u_{nd}\right), \tag{5.3a}$$

$$\sigma_{sd} = (\alpha-2)\frac{\partial u_{zd}}{\partial z}-\varepsilon^{\frac{1}{2}}(\alpha-2)\frac{\partial u_{nd}}{\partial \rho}+\varepsilon\frac{\alpha}{r}\left(\frac{\partial u_{sd}}{\partial s}-\frac{1}{a}u_{nd}\right), \tag{5.3b}$$

$$\sigma_{zd} = \alpha\frac{\partial u_{zd}}{\partial z}+\varepsilon^{\frac{1}{2}}(\alpha-2)\frac{\partial u_{nd}}{\partial \rho}+\varepsilon\frac{\alpha-2}{r}\left(\frac{\partial u_{sd}}{\partial s}-\frac{1}{a}u_{nd}\right), \tag{5.3c}$$

$$\tau_{nsd} = \varepsilon^{\frac{1}{2}}\frac{\partial u_{sd}}{\partial \rho}+\frac{\varepsilon}{r}\left(\frac{\partial u_{nd}}{\partial s}+\frac{1}{a}u_{sd}\right), \tag{5.3d}$$

$$\tau_{szd} = \frac{\partial u_{sd}}{\partial z}+\frac{\varepsilon}{r}\frac{\partial u_{zd}}{\partial s}, \tag{5.3e}$$

$$\tau_{nzd} = \frac{\partial u_{nd}}{\partial z}+\varepsilon^{\frac{1}{2}}\frac{\partial u_{zd}}{\partial \rho}. \tag{5.3f}$$

Substituting the composite expansion (5.1) into the boundary conditions (2.4) and (2.5) we obtain the following boundary conditions for the diffusion expansion:

$$\sigma_{zd} = \tau_{nzd} = \tau_{szd} = 0, \tag{5.4}$$

at $z = \pm 1$ for $0 \le \rho < \infty, 0 \le s \le L$ and $0 \le t < \infty$;

$$\sigma_{nd} = \varepsilon^2 f^{(2)}-\sigma_{ni}-\sigma_{nl}, \tag{5.5a}$$

$$\tau_{nsd} = \varepsilon^2 g^{(2)}-\tau_{nsi}-\tau_{nsl}, \tag{5.5b}$$

$$\tau_{nzd} = \varepsilon^3 h^{(3)}-\tau_{nzi}-\tau_{nzl}, \tag{5.5c}$$

at $\rho = 0$, for $0 \le s \le L$, $-1 \le z \le 1$ and $0 \le t < \infty$.

Similarly the initial conditions are

$$\mathbf{u}_d = \frac{\partial \mathbf{u}_d}{\partial t} = 0, \tag{5.6}$$

for $t = 0$ everywhere in the plate. Since we expect the bending effects considered in this section to be important only near the edge $\rho = 0$, as well as introducing an approximate semi-infinite domain for ρ, we introduce the decay condition

$$\lim_{\rho\to\infty} \mathbf{u}_d = 0, \tag{5.7}$$

for $0 \le s \le L$, $-1 \le z \le 1$ and $0 \le t < \infty$.

Now following the standard procedure an expansion for $\mathbf{u}_d$ in the form of a power series in $\varepsilon^{\frac{1}{2}}$ is assumed: $\mathbf{u}_d = \varepsilon\mathbf{u}_d^{(1)}+\varepsilon^{\frac{3}{2}}\mathbf{u}_d^{(\frac{3}{2})}+\cdots$. Here the expansion has a leading term of $O(\varepsilon)$ in order to match the prescribed edge conditions. Equations (5.2)–(5.4) can now be used to systematically derive the general form of the term $\mathbf{u}_d^{(k)}$, $k = 1, \frac{3}{2}, 2, \ldots$. The leading terms are:

$$u_{nd} = -\varepsilon^{\frac{3}{2}} z \frac{\partial U_{zd}^{(1)}}{\partial \rho} - \varepsilon^2 z \frac{\partial U_{zd}^{(\frac{3}{2})}}{\partial \rho} + \cdots, \quad u_{sd} = -\varepsilon^2 z \frac{\partial U_{zd}^{(1)}}{\partial s} + \cdots,$$
$$u_{zd} = \varepsilon U_{zd}^{(1)} + \varepsilon^{\frac{3}{2}} U_{zd}^{(\frac{3}{2})} + \varepsilon^2 \left(\frac{z^2}{2} \frac{\alpha - 2}{\alpha} \frac{\partial^2 U_{zd}^{(1)}}{\partial \rho^2} + U_{zd}^{(2)} \right) + \cdots, \tag{5.8}$$

where $U_{zd}^{(1)}$ is a function of ρ, s and t satisfying the equation

$$\frac{4(\alpha-1)}{3\alpha} \frac{\partial^4 U_{zd}^{(1)}}{\partial \rho^4} + \frac{\partial^2 U_{zd}^{(1)}}{\partial t^2} = 0. \tag{5.9}$$

Similar equations govern the behaviour of higher order terms $U_{zd}^{(k)} 1k = \frac{3}{2}, 2, \frac{5}{2}, \ldots$ occurring in (5.8). In all these equations the variable s only appears as a parameter, however in the right hand sides of the equations for $U_{zd}^{(k)}$, $k \geq 2$, derivatives with respect to s of the lower order terms occur. Therefore the expansion for $\mathbf{u}_d$ can only be extended for as many terms as smoothness with respect to s allows. In fact with the boundary conditions which we derive later and the initial conditions below, it can be shown that in order to obtain all terms up to $U_{zd}^{(N)}$ we require the data prescribed at $\rho = 0$ to possess $N-1$ derivatives with respect to s which are uniformly $0(1)$ in comparison with ε.

In order to solve for $U_{zd}^{(1)}$ initial conditions and boundary conditions are required. Using the initial quiescence condition (5.6) we deduce the initial conditions

$$U_{zd}^{(1)}(\rho, s, 0) = 0, \qquad \frac{\partial U_{zd}^{(1)}}{\partial t}(\rho, s, 0) = 0, \tag{5.10}$$

for $0 \leq \rho < \infty$, $0 \leq s \leq L$. We expect the boundary condition at $\rho = 0$ to involve the average bending effects, namely, bending moment and shear force. The boundary layer analysis of Section 6 leads to the adoption of the following boundary conditions for $U_{zd}^{(1)}$:

$$\frac{8(\alpha-1)}{3\alpha} \frac{\partial^2 U_{zd}^{(1)}}{\partial \rho^2}(0, s, t) = -\int_{-1}^{1} z\bar{f}^{(2)}(s, z, t)\, \mathrm{d}z, \tag{5.11a}$$

$$\frac{\partial^3 U_{zd}^{(1)}}{\partial \rho^3}(0, s, t) = 0, \tag{5.11b}$$

for $0 \leq s \leq L, 0 \leq t < \infty$. Similarly the boundary conditions for $U_{zd}^{(\frac{3}{2})}$ are:

$$\frac{8(\alpha-1)}{3\alpha} \frac{\partial^2 U_{zd}^{(\frac{3}{2})}}{\partial \rho^2}(0, s, t) = \frac{4(\alpha-2)}{3\alpha} \frac{1}{a} \frac{\partial U_{zd}^{(1)}}{\partial \rho}(0, s, t), \tag{5.12a}$$

$$\frac{8(\alpha-1)}{3\alpha} \frac{\partial^3 U_{zd}^{(\frac{3}{2})}}{\partial \rho^3}(0, s, t) = -\frac{1}{a}\int_{-1}^{1} z\bar{f}^{(2)}(s, z, t)\, \mathrm{d}z$$
$$- \frac{\partial}{\partial s}\int_{-1}^{1} z\bar{g}^{(2)}(s, z, t)\, \mathrm{d}z - \int_{-1}^{1} \bar{h}^{(3)}(s, z, t)\, \mathrm{d}z, \tag{5.12b}$$

for $0 \leq s \leq L, 0 \leq t < \infty$. Similar conditions hold for the higher order terms. Note that we have used the short-time quantities $\bar{f}^{(2)}$, $\bar{g}^{(2)}$ and $\bar{h}^{(3)}$ in the boundary conditions (5.11) and (5.12). The reasons for this procedure are best exhibited by an investigation of the validity of the assumption that the decay condition (5.7) holds uniformly for all time. It can easily be shown that solutions of equation (5.9) satisfying the initial conditions (5.10) and boundary conditions (5.11) do not decay as $\rho \to \infty$ for times as large as $0(1/\varepsilon)$. To overcome this difficulty we must match the expansion for u_d on to the long-time interior expansion u_l. This matching process (described in detail in a separate work [5]) is carried out by selecting the initial conditions for $\mathbf{u}_l$, say $\mathbf{u}_l = a^{(0)}+\varepsilon a^{(1)}+\cdots$, $\partial\mathbf{u}_l/\partial t = b^{(0)}+\varepsilon b^{(1)}+\cdots$ at $t = 0$, such that $\mathbf{u}_l$ and $\mathbf{u}_d$ are equivalent asymptotic expansions at some intermediate time scale, say $t^* = \sqrt{(\varepsilon)}t$. This matching procedure requires that the boundary data associated with the diffusion expansion decay with time. Therefore in this section we use only $\bar{f}^{(2)}$, $\bar{g}^{(2)}$ and $\bar{h}^{(3)}$ and require them all to be of order $0(1/t^{N+1})$ as $t \to \infty$. This allows the diffusion expansion to be extended up to $0(\varepsilon^N)$ provided that in addition these functions all possess N derivatives at $t = 0$ uniformly of $0(1)$ in comparison with ε.

6. BOUNDARY LAYER EXPANSION

In order to resolve the question of the nature of the solution near the edge of the plate and to decide which boundary conditions are appropriate for the equations governing the two interior expansions and the diffusion expansion we now investigate the boundary layer expansion we conjecture that for the gentle loading prescribed the detailed structure of the edge conditions is unimportant outside a boundary layer similar to that of the static analysis. To investigate this idea we introduce the boundary layer variable $\zeta = n/\varepsilon$. The equations of motion and the stress–displacement equations in terms of ζ, s, z and t can be readily obtained from equations (5.2) and (5.3) by substituting $\rho = \sqrt{(\varepsilon)}\zeta$ and therefore are not reproduced here. We now try a solution of the complete problem in the form

$$\begin{aligned}\mathbf{u}(x, y, z, t\,; \varepsilon) &= \mathbf{u}_i(x, y, z, t\,; \varepsilon)+\mathbf{u}_l(x, y, z, \tau\,; \varepsilon)\\ &\quad+\mathbf{u}_d(\rho, s, z, t\,; \varepsilon)+\mathbf{u}_b(\zeta, s, z, t\,; \varepsilon),\end{aligned} \tag{6.1}$$

where $\mathbf{u}_b$ represents the boundary layer correction.

The boundary conditions for the boundary layer correction are obtained by substituting the proposed solution (6.1) into the boundary conditions (2.4) and (2.5). These are:

$$\sigma_{nb} = \tau_{szb} = \tau_{nzb} = 0, \tag{6.2}$$

at $z = \pm 1$, for $0 \le \zeta < \infty$, $0 \le s \le L$ and $0 \le t < \infty$. The introduction of the semi-infinite interval for the variable ζ is consistent with the idea that the boundary layer term is unimportant except near the edge. Later we show that the errors resulting from this approximation are negligible. The other boundary conditions are

$$\sigma_{nb} = \varepsilon^2 f^{(2)} - \sigma_{ni} - \sigma_{nl} - \sigma_{nd}, \tag{6.3a}$$

$$\tau_{nsb} = \varepsilon^2 g^{(2)} - \tau_{nsi} - \tau_{nsl} - \tau_{nzd}, \tag{6.3b}$$

$$\tau_{nzb} = \varepsilon^3 h^{(3)} - \tau_{nzi} - \tau_{nzl} - \tau_{nzd}, \tag{6.3c}$$

at $\zeta = 0$, for $0 \le s \le L$, $-1 \le z \le 1$ and $0 \le t < \infty$ (the subscript b always indicates association with the boundary layer correction). From (2.6) we obtain the initial conditions

$$\mathbf{u}_b = \frac{\partial \mathbf{u}_b}{\partial t} = 0 \quad \text{for } t = 0, \tag{6.4}$$

$0 \le \zeta < \infty$, $0 \le s \le L$ and $-1 \le z \le 1$. In addition we impose the decay condition

$$\lim_{\zeta \to \infty} \mathbf{u}_b = 0 \tag{6.5}$$

for $0 \le s \le L$, $-1 \le z \le 1$ and $0 \le t < \infty$. This requirement is consistent with the conjecture that the boundary layer correction is only important near the edge of the plate.

We now assume a series representation for $\mathbf{u}_b$ in the form $\mathbf{u}_b = \varepsilon^2 \mathbf{u}_b^{(2)} + \varepsilon^{\frac{5}{2}} \mathbf{u}_b^{(\frac{5}{2})} + \cdots$ [the powers of $\varepsilon^{\frac{1}{2}}$ must be included since the boundary conditions (6.5) depend on powers of $\varepsilon^{\frac{1}{2}}$ arising from the diffusion expansion]. The lowest order problem obtained by substituting this expansion in the problem given above has the following governing equations:

$$\frac{\partial \tau_{nzb}^{(2)}}{\partial z} + \frac{\partial \sigma_{nb}^{(2)}}{\partial \zeta} = 0, \tag{6.6a}$$

$$\frac{\partial \tau_{szb}^{(2)}}{\partial z} + \frac{\partial \tau_{nsb}^{(2)}}{\partial \zeta} = 0, \tag{6.6b}$$

$$\frac{\partial \sigma_{zb}^{(2)}}{\partial z} + \frac{\partial \tau_{nzb}^{(2)}}{\partial \zeta} = 0. \tag{6.6c}$$

The corresponding boundary conditions are

$$\sigma_{zb}^{(2)} = \tau_{nzb}^{(2)} = \tau_{szb}^{(2)} = 0, \qquad \text{at } z = \pm 1; \tag{6.7}$$

and

$$\sigma_{nb}^{(2)} = f^{(2)} - \sigma_{ni}^{(2)} - \sigma_{nl}^{(2)} - \sigma_{nd}^{(2)}, \tag{6.8a}$$

$$\tau_{nsb}^{(2)} = h^{(2)} - \tau_{nsi}^{(2)} - \tau_{nsl}^{(2)} - \tau_{nsd}^{(2)}, \tag{6.8b}$$

$$\tau_{nzb}^{(2)} = -\tau_{nzi}^{(2)} - \tau_{nzl}^{(2)} - \tau_{nzd}^{(2)}, \tag{6.8c}$$

at $\zeta = 0$. From the decay condition (6.5) it follows that

$$\sigma_{nb}^{(2)}, \sigma_{sb}^{(2)}, \sigma_{zb}^{(2)}, \tau_{nzb}^{(2)}, \tau_{nsb}^{(2)}, \tau_{szb}^{(2)} \to 0 \quad \text{as } \zeta \to \infty. \tag{6.9}$$

In equations (6.9)–(6.12) the variables s and t appear only as parameters and in terms of ζ and z the problem is equivalent to a static elasticity problem for a semi-infinite elastic strip. An explicit solution for this problem cannot be readily obtained, however conditions necessary for the existence of a solution are easily deduced. These conditions, corresponding to self equilibrated end loads, are obtained by integrating the equations (6.6a, b, c) with respect to z between the limits ± 1, and respect to ζ between the limits 0 and ∞. After using the boundary conditions (6.7) and the decay conditions (6.9) we obtain the conditions

$$\int_{-1}^{1} \sigma_{nb}^{(2)} \, \mathrm{d}z = \int_{-1}^{1} \tau_{nsb}^{(2)} \, \mathrm{d}z = \int_{-1}^{1} \tau_{nzb}^{(2)} \, \mathrm{d}z = 0, \quad \text{at } \zeta = 0. \tag{6.10}$$

Two more conditions are obtained by multiplying (6.6a, b) by z and then integrating as above:

$$\int_{-1}^{1} z\sigma_{nb}^{(2)} \, \mathrm{d}z = 0, \quad \text{at } \zeta = 0, \tag{6.11a}$$

$$\int_{-1}^{1} \int_{0}^{\infty} \tau_{szb}^{(2)}(\zeta, s, z, t) \, \mathrm{d}\zeta \, \mathrm{d}z + \int_{-1}^{1} z\tau_{nsb}^{(2)}(0, s, z, t) \, \mathrm{d}z = 0. \tag{6.11b}$$

Using the boundary conditions (6.8a, b) and the results from the previous sections, the first two parts of (6.10) give

$$\sigma_{ni}^{(2)} = \frac{1}{2}\int_{-1}^{1} f^{(2)}\,\mathrm{d}z, \qquad \tau_{nsi}^{(2)} = \frac{1}{2}\int_{-1}^{1} g^{(2)}\,\mathrm{d}z, \tag{6.12, 13}$$

for $x, y\varepsilon\Gamma$. This provides a rational derivation of the boundary condition (3.5) previously adopted without justification. The last part of (6.10) is identically satisfied since $\tau_{nzi}^{(2)}$, $\tau_{nzl}^{(2)}$ and $\tau_{nzd}^{(2)}$ are all zero. Using the boundary condition (6.8a) in the condition (6.11a) we deduce the condition

$$\int_{-1}^{1} zf^{(2)}\,\mathrm{d}z = \int_{-1}^{1} z\sigma_{nl}^{(2)}\,\mathrm{d}z + \int_{-1}^{1} z\sigma_{nd}^{(2)}\,\mathrm{d}z, \tag{6.14}$$

for $x, y\varepsilon\Gamma$. From Sections 4 and 5 we know that the long-time effects must be associated with $\sigma_{nl}^{(2)}$ and the short-time effects with $\sigma_{nd}^{(2)}$. Therefore we take

$$\Sigma_{nd}^{(2)} = \frac{3}{2}\int_{-1}^{1} z\tilde{f}^{(2)}\,\mathrm{d}z, \qquad \Sigma_{nl}^{(2)} = \frac{3}{2}\int_{-1}^{1} z\hat{f}^{(2)}\,\mathrm{d}z \tag{6.15}$$

for $x, y\varepsilon\Gamma$. Here $\sigma_{nl}^{(2)} = z\Sigma_{nl}^{(2)}$ and $\sigma_{nd}^{(2)} = z\Sigma_{nd}^{(2)}$ and the Σ's are functions independent of z whose explicit forms in terms of $W_l^{(0)}$ and $U_{zd}^{(1)}$ respectively are easily obtained from the results of Sections 4 and 5. The first part of (6.15) provides the rational derivation of the boundary condition (5.11a) assumed previously without justification and the second part of (6.15) provides one of the boundary conditions required to solve equation (4.5) for $W_l^{(0)}$. Repeating the whole procedure for the $O(\varepsilon^{5/2})$ boundary layer equations we obtain more necessary conditions for solution which provide boundary conditions for higher terms in the expansion $\mathbf{u}_i$, $\mathbf{u}_l$ and $\mathbf{u}_d$. The boundary conditions (5.11b) and (5.12a) follow from these calculations.

One more boundary condition is required for the solution of equation (4.5) for $W_l^{(0)}$. In order to obtain this condition we must consider the $O(\varepsilon^3)$ boundary layer equations. At this order the equation comparable to the last part of equation (6.10) is

$$\int_{-1}^{1} \tau_{nzb}^{(3)}(0, s, z, t)\,\mathrm{d}z = \int_{-1}^{1}\int_{0}^{\infty} \left(\frac{\partial \tau_{szb}^{(2)}}{\partial s} - \frac{1}{a}\tau_{nzb}^{(2)}\right) \mathrm{d}\zeta\,\mathrm{d}z. \tag{6.16}$$

From the $O(\varepsilon^2)$ boundary layer equations it follows that $\int_{-1}^{1} \tau_{nzb}^{(2)}\,\mathrm{d}z = 0$ for all ζ, and then combining equations (6.11b) and (6.16) we obtain the result

$$\int_{-1}^{1} \tau_{nzb}^{(3)}(0, s, z, t)\,\mathrm{d}z + \frac{\partial}{\partial s}\int_{-1}^{1} z\tau_{nsb}^{(2)}(0, s, z, t)\,\mathrm{d}z = 0.$$

Using the boundary conditions (6.3b, c) this becomes

$$\int_{-1}^{1} h^{(3)}\,\mathrm{d}z + \frac{\partial}{\partial s}\int_{-1}^{1} zg^{(2)}\,\mathrm{d}z = \int_{-1}^{1} \tau_{nzl}^{(3)}\,\mathrm{d}z + \frac{\partial}{\partial s}\int_{-1}^{1} z\tau_{nsl}^{(2)}\,\mathrm{d}z$$
$$+ \int_{-1}^{1} \tau_{nzd}^{(3)}\,\mathrm{d}z + \frac{\partial}{\partial s}\int_{-1}^{1} z\tau_{nsd}^{(2)}\,\mathrm{d}z,$$

for $x, y \varepsilon \Gamma$. This relationship can be decomposed into its short-time and long-time parts to finally obtain

$$\int_{-1}^{1} \bar{h}^{(3)} \, dz + \frac{\partial}{\partial s} \int_{-1}^{1} z \bar{g}^{(2)} \, dz = \int_{-1}^{1} \tau_{nzd}^{(3)} \, dz + \frac{\partial}{\partial s} \int_{-1}^{1} z \tau_{nsd}^{(2)} \, dz, \tag{6.17a}$$

$$\int_{-1}^{1} \hat{h}^{(3)} \, dz + \frac{\partial}{\partial s} \int_{-1}^{1} z \hat{g}^{(2)} \, dz = \int_{-1}^{1} \tau_{nzl}^{(3)} \, dz + \frac{\partial}{\partial s} \int_{-1}^{1} z \tau_{nsl}^{(2)} \, dz, \tag{6.17b}$$

for $x, y \varepsilon \Gamma$. Conditions (6.17a) and (6.17b) provide the remaining boundary conditions required for the solution of the equation for $U_{zd}^{(\frac{3}{2})}$ and equation (4.5) for $W_l^{(0)}$ respectively. In fact (6.17a) leads directly to the boundary condition (5.12b). When rewritten in terms of physical variables the second part of condition (6.15) and condition (6.17b) can be recognized as the classical boundary conditions of Kirchhoff (see Love [3, p. 297]).

Other boundary conditions required for the determination of higher order terms in $\mathbf{u}_i$, $\mathbf{u}_l$ and $\mathbf{u}_d$ follow from the analysis of higher order boundary layer problems. The variables s and t only appear as parameters in the boundary layer problems and therefore the smoothness of the $O(\varepsilon^2)$ boundary layer term with respect to these variables is exactly the same as the data. In the higher order problems derivatives with respect to s and t of lower order terms appear as non-homogeneous terms in the governing differential equations. Therefore the boundary layer expansion can only be extended as far as this smoothness permits. However the restrictions already imposed in Section 3 require N continuous derivatives of $f^{(2)}$ and $g^{(2)}$ for $0 \le s \le L$ and $0 \le t < \infty$, therefore the boundary layer expansion can be extended to $O(\varepsilon^N)$. Also since $f^{(2)} = g^{(2)} = 0$ for $t = 0$ this smoothness requirement ensures that the boundary layer expansion satisfies the initial condition (6.4).

Following the work of Johnson and Little [6] it is straightforward to establish that the boundary layer corrections decay exponentially with ζ and therefore the decay condition (6.5) and the approximation of domain produce errors which are negligible.

7. SUMMARY

The combined expansion $\mathbf{u} = \mathbf{u}_i + \mathbf{u}_l + \mathbf{u}_d + \mathbf{u}_b$ formally satisfies all the conditions of the problem (2.2)–(2.6) up to $O(\varepsilon^N)$, where N is determined by the smoothness of the data. It remains to be proved rigorously that the expansion obtained in this way is a uniformly valid asymptotic approximation to the exact solution.

The leading terms in the two interior expansions satisfy equations identified as the classical equations of thin plates. The boundary layer analysis of Section 6 provides a derivation of the classical boundary conditions of thin plate theory. The higher order terms in the expansions can be regarded as thick plate corrections with the analysis presented here giving an indication of the limits of the applicability of classical dynamic thin plate theory in terms of the smoothness, in time and space, of the applied loads.

At the start of this work it was thought that the Timoshenko plate equation might arise naturally from the perturbation approach. However, the only procedure leading to this equation involved changing the usual algorithm of asymptotic analysis where the problem associated with each order of ε is treated separately. If two successive orders of ε are grouped together then the Timoshenko equation is obtained as the equation governing the leading term in a modified type of interior expansion. The general usefulness of this latter procedure seems limited and it was not studied further.

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Абстракт—Получаются уравнения ириближенных теорий, для динамики тонких упругих пластинок, путём применения формальных способов асимитотического разложения, пременимых в точной трехмерной теории динамической упругости. Определяются обыкновенные граничные условия для приближенных теорий, исходя из анализа граничного и диффузионного слоев.